1. (a) If $\sum_n a_n$ and $\sum_n b_n$ absolutely converge, does $\sum_n a_n b_n$ absolutely converge?
   (b) If $\sum_n a_n$ and $\sum_n b_n$ converge, does $\sum_n a_n b_n$ converge?

2. (a) If $\sum_n a_n$ converges, does $\sum_n (-1)^n a_n$ converge?
   (b) If $\sum_n a_n$ absolutely converges, does $\sum_n (-1)^n a_n$ absolutely converge?

3. Determine if the following series conditionally converge, absolutely converge, or diverge.

   (a) $\displaystyle\sum_{n\geq 1} \frac{(-1)^n (3n)! 3^n}{(4n)!}$

   (b) $\displaystyle\sum_{n\geq 1} \frac{\cos(n\pi)}{\log(\log n)}$

4. Which values of r does the following series converge?

$$\sum_n \frac{1}{n}\left(\frac{r+1}{r}\right)^n.$$

Solution.

1. (a) **TRUE**

   Since $\sum_n a_n$ converges we have $\lim_{n\to\infty} a_n = 0$ by the divergence test. So there is a N large enough such that when $n \geq N$ we have
   $$|a_n| < 1.$$
   So when $n \geq N$ we have
   $$0 \leq |a_n b_n| = |a_n||b_n| < |b_n|.$$
   Since $\sum_n b_n$ absolutely converges we have $\sum_n |b_n|$ converges, and thus by comparison test, $\sum_n |a_n b_n|$ converges. So $\sum_n a_n b_n$ converges absolutely.

   (b) **FALSE**

   Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. Since $\frac{1}{\sqrt{n}}$ is decreasing and converges to 0, we have that $\sum_n a_n, \sum_n b_n$ converge by the alternating series test. However,
   $$\sum_n a_n b_n = \sum_n \frac{(-1)^{2n}}{n} = \sum_n \frac{1}{n} = \infty.$$

2. (a) **FALSE**

   Let $a_n = \frac{(-1)^n}{n}$. Just as in 1. (b) we have by $\sum_n a_n$ converges by alternating series test. But,
   $$\sum_n (-1)^n a_n = \sum_n (-1)^n \frac{(-1)^n}{n} = \sum_n \frac{1}{n} = \infty.$$

   (b) **TRUE**

   Lets verify that $\sum_n (-1)^n a_n$, absolutely converges. We have,
   $$\sum_n |(-1)^n a_n| = \sum_n |a_n|.$$
   Since $\sum_n a_n$ absolutely converges, we have $\sum_n |a_n|$ converges, and thus $\sum_n (-1)^n a_n$ absolutely converges.

3. (a) Let's apply the ratio test.

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(3(n+1))!3^{n+1}}{(4(n+1))!}\Big/\frac{(-1)^n(3n)!3^n}{(4n)!}\right| \\
&= \lim_{n\to\infty}\frac{3^{n+1}}{3^n}\frac{(3n+3)!}{(3n)!}\frac{(4n)!}{(4n+4)!} \\
&= \lim_{n\to\infty}\frac{3(3n+1)(3n+2)(3n+3)}{(4n+1)(4n+2)(4n+3)(4n+4)} \\
&= \lim_{n\to\infty}\frac{3(3n+1)(3n+2)(3n+3)}{(4n+1)(4n+2)(4n+3)(4n+4)}\cdot\frac{n^{-4}}{n^{-4}} \\
&= \lim_{n\to\infty}\frac{3\frac{3n+1}{n}\frac{3n+2}{n}\frac{3n+3}{n}\cdot\frac{1}{n}}{\frac{4n+1}{n}\frac{4n+2}{n}\frac{4n+3}{n}\frac{4n+4}{n}} \\
&= \lim_{n\to\infty}\frac{3(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})\cdot\frac{1}{n}}{4+\frac{1}{n})(4+\frac{2}{n})(4+\frac{3}{n})(4+\frac{4}{n})} \\
&= \frac{3(3+0)(3+0)(3+0)\cdot 0}{(4+0)(4+0)(4+0)(4+0)} \\
&= 0 \\
&< 1
\end{aligned}
$$

So by ratio test we have the series converges absolutely.

(b) We first note that $\cos(n\pi) = (-1)^n$. So we can rewrite the series as the alternating series,

$$\sum_n \frac{(-1)^n}{\log(\log n))}.$$

Since $\log(\log n)$ in an increasing function and goes to infinity as n goes to to infinity we have $\frac{1}{\log\log n}$ is a decreasing function that converges to 0 (Prove it!). So have that by the alternating series test, the series converges.

We still need to determine if it absolutely converges, meaning we need to determine the convergence of:

$$\sum_n \frac{1}{\log\log n}.$$

We note that since $\log n \leq n$ we have

$$\log(\log n) \leq \log n \leq n.$$

By looking at the reciprocals we get,

$$\frac{1}{\log\log n} \geq \frac{1}{n}.$$

Since $\sum_n \frac{1}{n}$ diverges, we have our series does not absolutely converge. Thus our series conditionally converges.

4. Let us apply the ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\left(\frac{r+1}{r}\right)^{n+1} \bigg/ \frac{1}{n}\left(\frac{r+1}{r}\right)^{n}\right|$$
$$= \lim_{n\to\infty}\frac{n}{n+1}\left|\frac{r+1}{r}\right|$$
$$= \left|\frac{r+1}{r}\right|$$

We have the the series converges absolutely when $\left|\frac{r+1}{r}\right| < 1$, or when $|r+1| < |r|$. We have this is true when $r < -\frac{1}{2}$ (Prove it!).

Again by ratio test we have the series diverges when $\left|\frac{r+1}{r}\right| > 1$ or when $|r+1| > |r|$. We have that this is truw when $r > -\frac{1}{2}$.

So we know what happens to the series when $r \neq -\frac{1}{2}$. So lets compute. When $r = -\frac{1}{2}$ we have,

$$\sum_n \frac{1}{n}\left(\frac{-\frac{1}{2}+1}{-\frac{1}{2}}\right)^n = \sum_n \frac{(-1)^n}{n}.$$

This is the alternating harmonic series and converges by alternating series test. In conclusion we have the series converges when $r \leq -\frac{1}{2}$.